OBSTACLES IN THE EVOLUTION OF SECONDARY SCHOOL STUDENTS' MENTAL MODELS OF REASONING ON DECISION MAKING

Ana Serradó Bayés
Colegio La Salle-Buen Consejo, Spain
ana.serrado@gm.uca.es

A longitudinal study on the evolution of the mental levels of reasoning on decision-making situationally-provoked by a game of chance task to minimize the risk to lose is presented. The task was firstly implemented to 48 Spanish secondary-school students (age 12). Four years later, it was implemented again to 28 of these 48 students (age 15). A retrospective analysis was performed to identify the stochastic objects involved in the ways of thinking that helped to make decisions and students' mental levels of reasoning. Students mental levels of reasoning on decision-making evolved from uni-structural responses based on personal preferences to initiate the evolution to extended abstract responses. Three main obstacles constricted a further evolution: the deterministic nature given to the random generator, the lack of proportional reasoning and the ignorance of the relationship between the classical a priori and the frequentist model of probability.

INTRODUCTION

Over the last decade several studies have advanced in the understanding of what decisions are, how decision-making skill is acquired and its acquisition measured, providing a corpus of knowledge for the decision science (Cokely et al., 2018). From the many fields that decision science comprises, the interest of this paper falls on the complex integrative cognitive view that the philosophical, educational and stochastic knowledge can provide for analysing the dynamical nature of the decision-making process. This dynamical nature is analysed in this paper from two points of view. The first one states that the decision-maker sequentially samples evaluations based on partial cognitive models of decision-making for preferences that estimate the utility of an action until the preference for one action exceeds a threshold. The second one affirm that there is also an evolution in the cognitive models that the person develops for decision making (Busemeyer, 2015). In accordance with these two points of view, the interest of the paper remains on studying the dynamical nature of the decision-making process to minimize the risk to lose when Secondary school students play a game of chance.

The game of chance, named Integer Addition Bingo [IAB], consists in completing a winning card arrangement filled with ten integers from -10 to 10. An applet randomly generates two integers from -5 to 5, which students mentally must add. If the number result of the addition is on their card, they mark the result on it. The first student who have marked all the numbers of his/her card wins the game.

A task was constructed to research on students' reasoning on the decisions made to select some proposed card arrangements and construct their own card arrangements in order to minimize the risk to lose. The task consisted in a sequential process of playing with the IAB game of chance, doing mathematics, and reasoning about the situation in three differentiated phases. The first phase aimed to familiarize students with the game, and to trigger subjective intuitions about randomness. A second phase students had to select between some card arrangements to confront students with the difference between events and sequences of events and identifying that some events are more likely to occur. In the third phase, the students were asked to construct some card arrangements, analyse the distribution of frequencies when playing with those card arrangements and reasoning about the criteria used to construct the card to minimize the risk to lose.

In this paper, a retrospective analysis of students stochastic thinking and reasoning is presented to identify the evolution of students' mental models of reasoning on decision making and discern the obstacles that difficult a further evolution.

OBSTACLES IN THE EVOLUTION OF DECISION-MAKING

The evolution of decision science has been parallel to the evolution of the probabilistic science (Cokely et al., 2018). Two historic milestones have been essential to understand the basis of this parallel evolution and to frame a model for studying the evolution of secondary school students' mental models of reasoning on decision making.

In: S Budgett (Ed.), *Decision Making Based on Data*
Proceedings of the Satellite conference of the International Association for Statistical Education (IASE),
August 2019, Kuala Lumpur, Malaysia. iase-web.org/Conference_Proceedings.php

The first historic milestone is related with the seminal works of Blaise Pascal and Pierre Fermat on the gambling problem in 1956. Those works provided of a logical system that connected the probability theory and the decision-making in bets in games of chance (Hacking, 2006). This interconnected logic system between probabilistic and decision making theory has been taken in consideration in the design of the IAB task, which aims to evolve decisions made by the students to select and construct their card arrangements to win through the analysis of the probabilities of appearance of each pair of added numbers. Those decisions may be done based on their personal preferences of the card arrangements or based on the logic system of probabilities that they develop. Nevertheless, Serradó (2019) concluded that a deterministic way of thinking may hinder the evolution of the personal decisions to decisions rationally bounded in classical probabilistic postulates. A second historic milestone considered in this paper is the *theorema aureum* of Jakob Bernoulli –currently known as "law of large numbers". This theorem informs about the possibility of connecting probability to relative frequencies. Despite the potentiality of this theorem to experimentally measure probabilities, Bernoulli recognised two obstacles (Borovcnik & Kapadia, 2014).

A first obstacle is related with the insufficiency of data to measure the probability. This epistemic obstacle is recognised as a way of thinking of some subjects, who do not consider the need of sufficient data to conclude on the experimental probability of a situation. Consequently, this way of thinking may prone wrong decisions based in insufficient data to evaluate the situation on hand. A second obstacle is the insufficiency of the condition of independence of the events to conclude about the experimental probability. Educational research on this field informs, that subjects must understand the in/dependence of the results. A complete understanding of this in/dependence of events needs to internalize and distinguish the meaning of different stochastic objects, such as the events and the outcomes (Savard, 2014). They also need to understand the proportional relationship of the quantities of the favourable and possible outcomes (Saldanha & Liu, 2014). Furthermore, there is also the need of understanding the relationship between a classical a priori and a frequentist model of probability in estimating the probability of random outcomes.

We also have considered this challenge when designing the IAB task. This means that students are enhanced in thinking on the relationship between the classical a priori and the frequentist model of probability underlying the Integer Addition Bingo game of chance task. Furthermore, they are asked to make decisions to select and construct the card arrangements according with their understanding of the information that these models provide. We hypothesize that thinking in this relationship could help to surpass the obstacles described by Bernoulli and, consequently, may rationally bound their decisions under the recognition of the probabilities known.

From a philosophical point of view, this recognition needs to capture the essence of the duality between measurability/immeasurability, objectivity/subjectivity of probability and the surability/insurability of the probabilistic outcomes (Knight, 1921). Knight goes further an affirms that under the essence of the duality opinion/action (thinking/deciding), there is neither ignorance nor complete and perfect information about the uncertainty of the situation. When the decisions are done with a partial knowledge of the situation, Knight (1921) speaks of decisions under uncertainty. Meanwhile, when all the alternative possibilities are known and the probability of the occurrence of each event can be accurately ascertained, he named those as decisions of risk.

The description of these two milestones leads to present a scheme that informs about the historic evolution of the notion of decision-making. Moreover, it has provided links between models of probability and models of decision-making. Summing up, on the one end of this schema, there are the decisions based on personal preference; on the other end, there are the globally rational bounded decisions made under uncertainty –if probabilities are unknown- and decisions of risk –if the probabilities are known (Serradó, 2019).

MENTAL MODELS OF REASONING ON STOCHASTIC DECISION MAKING CONTEXTS

Although this schema provides a theoretical framework of the historic evolution of the notion of decision-making, students cannot be ascribed to those general cognitive mental levels of reasoning on decision-making. In fact, they may show situational mental levels of reasoning –that is, in the context of a specific task (Serradó, 2018). In general, this author expresses that a theory of mental models is based on a structural and a dynamic aspect of thinking: (a) a way of thinking as a structural static aspect of thinking, which involves recognising the physical objects in a given situation and their characteristics; (b) a way of understanding as a functional dynamic aspect of thinking, which refers to the construction of a mental model when coping with demands of a specific situation. In consequence, the mental models can be changed, enriched or modified while a student is working on representations of a situation; and mental models may show interpersonal differences in their structural and functional aspects. Traditionally, researchers have used the SOLO (Structure of Observed Learning Outcome) model of Biggs and Collis (1982) when coding responses to describe the mental models of subjects. In the field of stochastics, the SOLO taxonomy has been used in many prominent frameworks to analyse the development of probabilistic thinking (Mooney, Langrall, & Hertel, 2014). Those authors highlight that, in general, students' probabilistic thinking moves from being idiosyncratic to proportional in nature.

Particularly, Serradó (2018) identified four of this five mental levels of reasoning on decision making situationally-provoked as a result of students engagement in the IAB decision-making game of chance task: (a) *pre-structural responses:* decisions in this category where based on personal preferences; there were no logical connection to the understanding of the uncertainty, randomness, or hazard involved in the IAB task; (b) *uni-structural responses:* decisions in this category were said to be made "under uncertainty" and were rationally bounded in the analysis of the modal clumps of the distribution of the relative frequencies; (c) *multi-structural responses:* decisions in this category were either made "under uncertatinty" (if they are rationally bounded on reasoning on the events associated with the majority of the data) or were decisions "of risk" (if they were rationally bounded on reasoning about the density and symmetry of the classical theoretical distribution of probabilities); and (d) *relational responses:* decisions in this category can be classified as both "under uncertainty" (reasoning rationally bounded on the analysis of the majority of the data in the distribution of the relative frequencies) and "of risk" (rationally bounded on reasoning on the symmetry of the classical theoretical distribution of probabilities). The evolution to a higher mental level of reasoning on decision-making with extended abstract responses was constricted by a deterministic way of thinking, the difficulties of discerning between the randomness of the generator, the randomness of the events and sequences of events, and a lack of previous knowledge about the measures of centre for frequency distributions (Serradó, 2018).

LONGITUDINAL DESIGN BASED RESEARCH

In 2015, a Design Based Research (DBR), grounded on theory and empirical products (Engeström, 2011), was initiated with the IAB task design (described in the introductory section). The IAB task was implemented to 48 Grade 7 students (ages 12-14) in a Spanish middle school located in a low socio-economic coastal city. Due to the constrictions described in the previous section about the randomness of the events and sequences of events, in 2016 a second cycle of DBR was initiated improving the IAB task design. On the one hand, the improvement consisted in including questions about: (a) the differences between the numbers randomly generated by the IAB random generator and the random events obtained by the addition of the random-generated numbers, (b) the strategies to discriminate the independence of the events and of the sequences of events. On the other hand, the task was improved to engage students in a deliberate dialogue about the uncertainty of the pseudo-randomly generated numbers. The improved task was implemented again to 28 of these 48 students, who participated in the first cycle of DBR to longitudinally study the evolution of students' mental levels of reasoning in decision-making. Four cases of evolution are highlighted in the next section and summarised in Table 1.

Table 1. Students evolution and obstacles that emerged

S	*2015 evolution level*	*2018 evolution level*	*Obstacle*
Ta	Uni-structural	Multi-structural	Deterministic way of thinking
A	Initiating a relational	Relational	Lack of proportional thinking
N	Multi-structural	Initiating an extended abstract	Ignorance of the relationship between the theoretical and the frequentist model of probability

RESULTS AND DISCUSSION

Student Ta, concluded that the IAB helped her to learn about: “the values of the relative frequency. *The total of the relative frequency that is the probability, and the equal probable outcomes”* (Ta 2015). Ta reasoning highlights a misunderstanding of the relationship between a frequentist model and a theoretical classical model of probability. Moreover, when she stated that the probability is the total of the relative frequency, she presents a misundertanding of the meaning of the law of large numbers. The previous theoretical analysis has provided information about two possible obstacles recognised by Bernoulli in relation with the law of large numbers (Borovcnik & Kapadia, 2014). One of them is the insufficience of the data to measure the values of the relative frequency that the student Ta declares that she has learned about.

In the second cycle of DBR initiated in 2017, the student played three times with the IAB, before making the first decisions about how to construct a card to minimize the risk to lose. The second retrospective analysis of Ta (aged 16) responses about her decisions to construct a card to minimize the risk to lose is presented. She argued: *“thinking on the numbers with higher probability of appearance and the ones that have repeated more before”* (Ta 2018). Her responses are based on the probability and frequency of appearance after three plays with the IAB. In this three plays, the student Ta analysed a sequence of 66, 55 and 41 events respectively. In consequence, Ta’s stochastic way of thinking is based on her understanding on how short-term behaviour of the relative frequencies distribution is sufficient to estimate the probability. In words of Borovcnik and Kapadia (2014) based on the ideas of Bernoulli an obstacle to understand the law of large numbers may emerge. Moreover, Serradó (2019) conjectured that those subjects who base their perceptions in short-term behaviour of the frequencies may wrongly and intuitively think that an experiment which is random has a unique formulation. To deepen in this conjecture, a retrospective analysis of the deliberate dialogues in which Ta participated was performed.

130	F	I think that the game is random. Because the addition, you always know that two plus two is four. The probability is of the addition.
131	T	[Teacher interrupts the students] I have asked about the randomness and not the probability.
132	F	Because every [ball] has the same… the same… Ummm!
133	ML.	I think the addition of the [value] of the two balls.
134	T	You think that the addition is random, why?
135	ML.	Because it is what it is going to appear.
136	Ta	Random is the card that you select, because the numbers that you write on the card are the numbers that you want to appear. And, the ones that…

The student Ta was unable conclude her mental action of interpreting the situation and understanding the random nature of the IAB pseudo-generator. Nevertheless, after this dialogue when asked to individually reason on her decisions to construct the cards, she argued: *“Those that I think can come out with more certainty, because they are closer to the theoretical probability. [...] And, you do not have the certainty that the same number always comes out”* (Ta 2018). The student uses in her reasoning twice the word *certainty*. May be she views the world as being connected though cause and effect. Saldanha and Liu (2014) interpret that in this case probability is considered a model that is choosen for a certain situation with the purpose of approximating phenomena and gaining information. This choice is founded upon a realization that the model is expected to be more powerful predictor of outcomes over the long run than a deterministic analysis. This is not the case of the student that never uses information about long-runs. Coherently, the way of thinking of the student can be understood as deterministic, being the deterministic nature given to the IAB pseudo-generator a possible obstacle for internalizing the relationship between two stochastic objects, events and results (Savard, 2014).

Summing up, an according to the SOLO taxonomy (Biggs & Collis, 1982) and the synthesis about students’ probabilistic thinking (Mooney et al., 2014), the mental model of the Ta student in the first cycle of the DBR evolved from a pre-structural to a uni-structural level of responses. In this second cycle of DBR, her mental models of reasoning have evolved from a uni-structural to multi-structural level of responses, because she recognises the potentiality of the frequential model and classical model for making predictions and, coherently, execute the decisions. Even though, she is unable to consolidate her idiosyncratic stochastic way of thinking due to the deterministic nature given to the IAB pseudo-random generator as a way of misunderstanding the long-run events.

Returning to Saldanha and Liu (2014) arguments on the evolution from deterministic ways of thinking to probabilistic ones considering the power of the probability as a predictor of outcomes, the student A in the first cycle of DBR in 2015 (aged 12) argued about this predictive potential of the probability. She initally evolved form pre-structural to relational level of responses. She reasoned about the modal clumps, the symmetry of the distribution of probabilities, the density of the frequencies and she was able to relate different aspects of the distribution and understand the difference between probability and frequence (*"one is what could happen and the other is what happened"* (A 2015). During the second cycle of DBR her individual responses highlight a possible evolution in the understanding of the value of increasing the number of runs. Her response was: *"If it had made more runs, the relative frequency would have increased and it would be the [distribution] more symmetric to the theoretical probability".* According to the SOLO taxonomy (Biggs & Collis, 1962), the responses of the student A could be codified as relational. She was able to integrate to her stochastical thinking model the predictive nature of the IAB random generator and the need of more runs to conclude about the predictive nature of the experimental frequentist model of probability. The statement about the increment of the relative frequency informed that she still lacks of a proportional thinking (Mooney et al., 2014). In the second implementation of the task (2018), the lack of proportional thinking of was identified in other two students when dialoguing about what could happen when increasing the number of runs.

271	T	Let's our mind fly! Think about what could happen if instead of 66 throws, we would have 350. I know that Fran is the winner, but that the game would have needed 350. What do you think it could have happened?
272	JR.	Would the zero have appeared more times?
273	T	That the zero could have appeared more times.
274	C	That the relative frequency would have been smaller, because if you make a quotient with more numbers. This must be smaller.
275	T	But, he says that it would appear more times.
276	N	Then it would be bigger.
277	T	Would it be bigger or not?

The dialogue highlights that the students C and N have difficulties in capturing the essence of this proportional reasoning. Nevertheless, they were able to conjecture about the law of large numbers. Different levels of argumentation were found. The student N (2018) argued, exclusively, analysing the smotheness of the relative frequency distribution (*"how many more times it repeats, it will tendo to stabilize more"*). In her response about how to construct the card to minimize the risk to loose, she used both the theoretical probability distribution and the stabilized relative frequency distribution (*"the symmetry of the theoretical probability and the modal clumps of the stabilized relative frequency distribution"* N, 2018). The student C (2018) response added to the student N initial perception of the law of large numbers the idea of approximation to the theoretical classical probability (*"When it starts to be very stabilized, the difference will be less appreciated, but it will continue to stabilize. The relative frequency will never be equal to the theoretical probability, but it stabilize the values"* C, 2018). Although she was able to advance in the understanding of the law of large numbers, she was not able to transfer her way of thinking into the construction of the card to minimize the risk to lose. She wrote: *"I will use numbers close to the median [of the stabilized relative frequency distribution], because I do not know exactly the values that are going to appear".*

Accordingly to the SOLO taxonomy (Biggs & Collis, 1982), those responses show an evolution to an extended abstract response where decisions are based on the evaluation of the probabilities on hand. Students have developed the first insights about the relationship between the stabilized relative frequencies distribution and the theoretical distribution. Nevertheless, we believe that the students have not been able to transfer this understanding to an extended abstraction of how the simulation of sequences of events using the theoretical model lead to evaluate more cases and made consequently decisions of risk.

CONCLUSION

From a theoretical point of view, the design based research developed has provided insights about how historically the evolution of the theory of decision making and probability have been parallel. The study of two milestones in this historical evolution have allowed identifying possible epistemic obstacles in the students' decision-making. This theoretical framework has been summarized in a scheme that goes from decisions based on personal preferences to decisions rationally bounded in the uncertainty of the situation –if the probabilities are unknown, and of risk –if the probabilities are known. This theoretical framework has been used for a retrospective and longitudinal analysis of the mental levels of reasoning on decision-making when situationally-provoked by a game of chance task. A student, whose responses in the first implementation of the task evolved from a pre-structural to a uni-structural level, in the second implementation evolved to a multi-structural level. An obstacle for evolving to a higher level of response was the deterministic way of thinking about the nature of the IAB pseudo-random generator. A student, whose responses in the first implementation of the task were classified as evolving to relational, was unable in the second implementation to arise this level. Her responses show a lack of proportional thinking that hindered the understanding of the relationship between the experimental and classical model of probability and difficult making decisions of risk. Finally, two students, whose initial level of response was clustered as multi-structural, in the second implementation of the task initiated the evolution to an extended abstract level. Further evolution was constricted by a lack of understanding of how the relationship stablished between the experimental frequentist model and the classical theoretical one could help them to evaluate the situation and make decisions of risk according to their evaluation. The determination of those mental levels of reasoning on decision making is situationally provoked by many variables (IAB game of chance selected, the task implemented or the design based research methodology used) that limit the application of the current theoretical findings in other contexts.

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